A Study of Differential Equations That Can Be Transformed Into Bessel's Equation

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Abstract.

In mathematics, the Bessel functions are the solutions to the differential Bessel's equation. It is considered the most widespread and important when n is an integer or a half integer.
At the beginning of this research, some important definitions are presented. There will be a definition of the differential Bessel equation, as well as the kinds of Bessel functions (first kind and second kind), their recurrence relations and functions related to the Bessel functions, and some definitions that have relations to the research.
The paper aims at studying the differential equations transformed into Bessel's equation and how to solve it, find the general solution to it, and study when n is an integer or a half integer numbers for both kinds $J_n(x), Y_n(x)$.

Keywords: Bessel's differential equations, Generating function, Recurrence relations, Transformed.

1. Introduction.

Differential equations are the basis of all sciences especially engineering and applied sciences. They are among the most important branches of mathematics in the past and the present. The differential equations are also a basis for solving many issues in the fields of mathematical, physical and electrical sciences, as well as in the field of chemical and mechanical engineering where the mathematical problem is formulated into a differential equation and then solved in many ways that contribute to solving differential equations. Among these methods of solving differential equations is the differential Bessel's equation. It should be noted that our research is concerned with solving differential equations that are transformed to the Bessel's differential equation and how to solve it.

1-1. Definition.

the following differential equation (see [1] , [4]).

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad , \quad n\geq 0 \qquad (1)$$

is known as Bessel's equation of order n, and the general solution of Bessel's equation (1) of order n is.

$$y=C_1J_n(x)+C_2Y_n(x) \qquad (2)$$

The solution $J_n(x)$, which has a finite limit as x approaches zero, is called a Bessel function of the first kind and order n.
The solution $Y_n(x)$ which has no finite limit as x approaches zero, is called a Bessel function of the second kind and order n.
If the independent variable x in (1) is changed to λx where when λ is a constant, the resulting equation is:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0 \quad , \quad n\geq 0 \qquad (3)$$

With general solution (see [3] , [4]):

$$y=C_1J_n(\lambda x)+C_2Y_n(\lambda x) \qquad (4)$$

1-2. Bessel functions of the first kind.

We define the Bessel functions of the first kind of order n as (see [1] , [5]):

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^{n+2k}}{k!\,\Gamma(n+k+1)} \qquad (5)$$

Where $\Gamma(n+1)$ is the gamma function. If n is a positive integer, $\Gamma(n+1)=n!$, $\Gamma(1)=1\ for\ n=0$, (5) becomes (see Fig 1):

$$J_n(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^24^2}-\frac{x^6}{2^24^26^2}+\cdots \qquad (6)$$

If n is half an odd integer, $J_n(x)$ can be expressed in terms of sines and cosines.

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\sin x \quad , \quad J_{\frac{-1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cos x \qquad (7)$$

A function $J_{-n}(x)$, $n>0$, can be defined by replacing n by $-$n in (5). If n is an integer then we can show that.

$$J_{-n}(x)=(-1)^nJ_n(x) \qquad (8)$$

If n is not an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly independent, and for this case the general solution of (1) is:

$$y=AJ_n(x)+BJ_{-n}(x) \qquad ,n\neq 0,1,2,3,\ldots \qquad (9)$$

1-3. Bessel functions of the second kind.

We shall define the Bessel functions of the second kind of order n as (see [1]).

$$Y_n(x)=\begin{cases}\dfrac{J_n(x)\cos(n\pi)-J_{-n}(x)}{\sin(n\pi)} & ,\ n\neq 0,1,2,3,\ldots\\ \lim_{p\to n}\dfrac{J_p(x)\cos(p\pi)-J_{-p}(x)}{\sin(p\pi)} & ,\ n=0,1,2,3,\ldots\end{cases} \qquad (10)$$

For the case where $n=0,1,2,3,\ldots$, we obtain the following series expansion for $Y_n(x)$.

$$Y_n(x)=\frac{2}{\pi}\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_n(x)-\frac{1}{\pi}\sum_{k=0}^{n-1}(n-k-1)!\left(\frac{x}{2}\right)^{2k-n}-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\emptyset(k)+\emptyset(n+k)\}\frac{(\frac{x}{2})^{2k+n}}{k!\,(n+k)!}$$

where $\gamma=0.5772156$ Be a Euler constant, $\emptyset(p)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}$, $\emptyset(0)=0$ (see Fig 2).

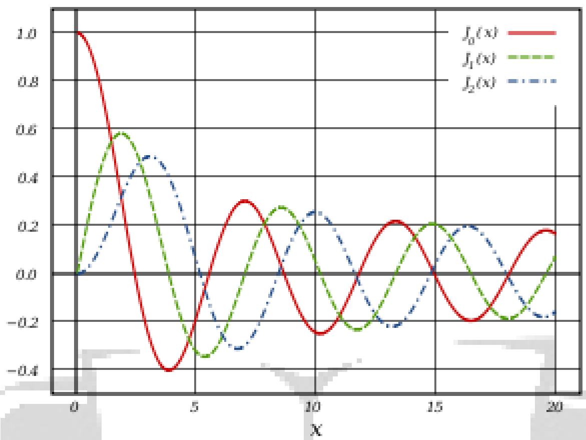

Figure 1: Bessel functions of the first kind, $J_n(x)$ for integer orders $n=0,1,2$

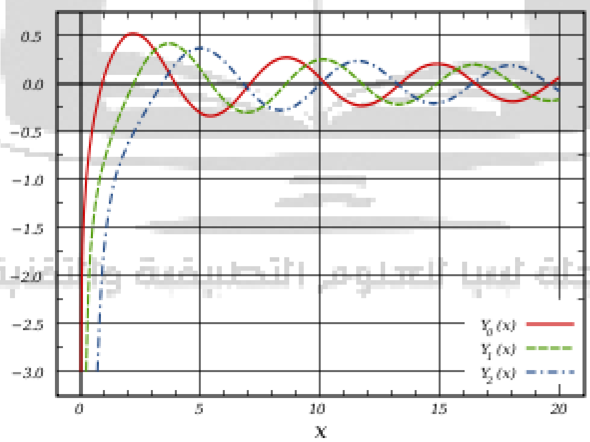

Figure 2: Bessel functions of the second kind, $Y_n(x)$ for integer orders $n=0,1,2$

2. Generating function for $J_n(x)$.

The function.

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \tag{11}$$

Is called the generating function for Bessel functions of the first kind of integer order.
It is very useful in obtaining properties of these functions for integer values of n which can then often be proved for all values (see [5]).

3. Recurrence Relations for $J_n(x)$ (see [3]).

The following results are valid for all values of n.

1) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

2) $\acute{J}_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

3) $x\acute{J}_n(x) = nJ_n(x) - xJ_{n+1}(x)$

4) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

5) $x\acute{J}_n(x) = xJ_{n-1}(x) - nJ_n(x)$

6) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

4. Functions related to Bessel functions (see [2] , [4]).

(4-a): Hankel Functions (Bessel functions of the third kind).

A third kind of functions (complex - valued) for $n \in \mathbb{R}, n \notin \mathbb{Z}$.

$$H_n^{(1)} = J_n(x) + iY_n(x) \quad , \quad H_n^{(2)} = J_n(x) - iY_n(x)$$

(4-b): Modified Bessel functions.

The modified Bessel functions of the first kind of order n are defined as follows:

$$I_n(x) = i^{-n} J_n(ix) = e^{\frac{-n\pi i}{2}} J_n(ix)$$

If n is an integer number then $I_{-n}(x) = I_n(x)$.

If n is not an integer number, then $I_{-n}(x) and\ I_n(x)$ are linearly independent.

(4-c): Kelvin Functions.

The Kelvin functions are obtained from the real and imaginary portions of this solution as follows.

$$ber_n(x) = Re\, J_n\left(i^{\frac{3}{2}} x\right)$$
$$bei_n(x) = Im\, J_n\left(i^{\frac{3}{2}} x\right)$$
$$J_n\left(i^{\frac{3}{2}} x\right) = ber_n(x) + i\, bei_n(x)$$
$$ker_n(x) = Re\, i^{-n} K_n\left(i^{\frac{1}{2}} x\right)$$
$$kei_n(x) = Im\, i^{-n}\, K_n\left(i^{\frac{1}{2}} x\right)$$
$$i^{-n}\, K_n\left(i^{\frac{1}{2}} x\right) = ker_n(x) + i\, kei_n(x)$$

These functions are useful for being related to the equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (ix^2 + n^2)y = 0 \quad , \quad n \geq 0 \qquad (12)$$

And the general solution is :

$$y = C_1 J_n\left(i^{\frac{3}{2}}x\right) + C_2 K_n\left(i^{\frac{1}{2}}x\right) \qquad (13)$$

If we put $\lambda^2 = i$ in equation (12), So we have the modified Bessel's equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\lambda^2x^2 + n^2)y = 0 \ \ , \ \ n \geq 0 \qquad (14)$$

And the general solution is :

$$y = AI_n(\lambda x) + BK_n(\lambda x) \qquad (15)$$

Example (1):

Solve the differential equation $x^2y'' + xy' + (3x^2 - 2)y = 0$ **?**

Solution.

This equation has order $\sqrt{2}$ and differs from the standard Bessel equation only by factor 3 before x^2. Therefore, the general solution of the equation is expressed by the formula $y(x) = C_1J_{\sqrt{2}}(\sqrt{3}x) + C_2Y_{\sqrt{2}}(\sqrt{3}x)$.

where C_1, C_2 are constants, $J_{\sqrt{2}}(\sqrt{3}x)$ and $Y_{\sqrt{2}}(\sqrt{3}x)$are Bessel functions of the 1st and 2nd kind, respectively.

Example (2):

Solve the differential equation $x^2y'' + xy' - (4x^2 + \frac{1}{2})y = 0$ **?**

Solution.

This equation differs from the modified Bessel equation by factor 14 in front of x^2. The order of the equation is $n = \frac{1}{\sqrt{2}}$. Then the general solution is written through the modified Bessel functions in the following way:

$y(x) = C_1I_{\frac{1}{\sqrt{2}}}(2x) + C_2K_{\frac{1}{\sqrt{2}}}(2x)$ where C_1, C_2 are arbitrary constants.

Now, after the introduction and previous definitions, we are now studying the general solution to differential equations that has transformed into the Bissell differential equation.

4. Equations transformed into Bessel's equation.

Any differential equation we can transformed to this equation.

$$x^2\frac{d^2y}{dx^2} + (2k+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0 \qquad (16)$$

where β, r, α, k are constants, has the general solution.

$$y(x) = x^{-k}\left[C_1J_{\frac{h}{r}}\left(\frac{\alpha x^r}{r}\right) + C_2Y_{\frac{h}{r}}\left(\frac{\alpha x^r}{r}\right)\right] \qquad (17)$$

where $h = \sqrt{k^2 - \beta^2}$, C_1, C_2 are constants (see [1]).

Example (1):

Obtain the general solution of $4y'' + 9xy = 0$ **?**

Solution.

In terms of Bessel functions.

Multiplying by $x^2$and dividing by 4, we have.

$$x^2y'' + \frac{9}{4}x^3y = 0$$

We can now identify with equation (16) , namely.

$x^2\frac{d^2y}{dx^2} + (2k+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0$ as follows:

$\beta = 0 \,, r = \frac{3}{2} \,, \alpha = \frac{3}{2} \,, k = \frac{-1}{2} \,, h = \frac{1}{2}$.

The general solution is $y(x) = \sqrt{x}\left[C_1J_{\frac{1}{3}}\left(x^{\frac{3}{2}}\right) + C_2J_{\frac{-1}{3}}\left(x^{\frac{3}{2}}\right)\right]$, where C_1, C_2 are constants.

Example (2):

Solve the differential equation $xy'' + y' + ay = 0$ **?**

Solution.

In terms of Bessel functions.

Multiplying by x, we have.

$$x^2y'' + xy' + axy = 0$$

We can now identify with equation (16) , namely.

$x^2\frac{d^2y}{dx^2} + (2k+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0$ as follows:

$\beta = 0 \,, r = \frac{1}{2} \,, \alpha = \sqrt{a} \,, k = 0 \,, h = 0.$

The general solution is $y(x) = C_1J_0(2\sqrt{ax}) + C_2Y_0(2\sqrt{ax})$, where C_1, C_2 are constants.

Example (3):

Solve the differential equation $4xy'' + 4y' + y = 0$ **?**

Solution.

In terms of Bessel functions.

Multiplying by x and dividing by 4, we have.

$$x^2y'' + xy' + \frac{1}{4}xy = 0$$

We can now identify with equation (16) , namely.

$x^2\frac{d^2y}{dx^2} + (2k+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0$ as follows:

$\beta = 0 \,, r = \frac{1}{2} \,, \alpha = \frac{1}{2} \,, k = 0 \,, h = 0.$

The general solution is $y(x) = AJ_0(\sqrt{x}) + BY_0(\sqrt{x})$, where A, B are constants.

Example (4):

A coil spring is such that a force of 8 lbs. will stretch the spring 6 $inches$. If the spring is suspended in a vertical position with a 4 lb. weight attached to the lower end (hanging free), and then the lower end is pushed up to a point 2 $inches$ above the point of equilibrium and released, determine the equation of motion. Use 32 ft/sec^2. for g.

Solution.

The differential equation is formulated as follows.

$$Force = (mass)(acceleration)$$
$$= \frac{d}{dt}\left(\frac{W}{g}\frac{dy}{dt}\right)$$

Since $\frac{W}{g}$ is a constant in this problem, we get by carrying out the indicated differentiation.

$$F = \frac{W}{g}\frac{d^2y}{dt^2}$$

This force is equated to the restorative force, which is the product of the spring constant k and the displacement y of the lower end from the point of equilibrium.
We have.

$$8\,lbs = k(stretch - distance\ in\ feet)$$

whence $k = 16$. Thus the restorative force is $-16y$. So, upon equating these two forces, we have the differential equation of motion (ignoring friction):

$$\frac{4}{32}\frac{d^2y}{dt^2} + 16y = 0$$

Or.

$$\frac{d^2y}{dt^2} + 128y = 0$$

In terms of Bessel functions.
Multiplying by t^2, we have.

$$t^2y'' + 128t^2y = 0$$

So that.
$\beta = 0\,, r = 1\,, \alpha = \sqrt{128}\,, k = \frac{-1}{2}\,, h = \frac{1}{2}$.
The general solution is $y(x) = \sqrt{t}\left[C_1J_{\frac{1}{2}}\left(\sqrt{128}\,t\right) + C_2J_{\frac{-1}{2}}\left(\sqrt{128}\,t\right)\right]$, where C_1, C_2 are constants.
With reference to equation No (7), we may write the general solution:
$y(x) = A\,Sin\left(\sqrt{128}\,t\right) + BCos\left(\sqrt{128}\,t\right)$, $where\ A = \sqrt{\frac{2}{\pi}}\,C_1$, $B = \sqrt{\frac{2}{\pi}}\,C_2$.

5. Conclusions.

We have noticed through our study that many differential equations can be transformed into Bessel's equation that were indicated in this paper with number (16), and therefore we can find the general solution for it with its first kind and second kind by following the mentioned steps.

6. Recommendations.

By studying the solution of differential equations that can be transformed into Bessel's equation which were indicated by number (16), researchers recommend expanding the study of these differential equations, especially when (h= complex number) to better determine the advantages of the solution.

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