

# Using Bernoulli's Differential Equation To Solve Thermodynamic And Aerodynamic Problems Describing Changes In Velocity Or Pressure

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## Abstract

Bernoulli's equation represents a cornerstone in fluid dynamics, describing the interdependence of pressure, velocity, and elevation along a streamline. Rooted in the principle of energy conservation, it offers essential insights into the behavior of thermodynamic and aerodynamic systems. This study investigates the application of Bernoulli's differential form to analyze variations in velocity and pressure within fluid flows. The derivation, underlying assumptions, and practical implications of the equation are discussed in detail, with emphasis on its use in aerodynamic analyses (e.g., lift and drag estimation) and thermodynamic applications (e.g., flow behavior in nozzles and diffusers). Through selected case studies and computational simulations, the utility of Bernoulli's equation in engineering design and system analysis is demonstrated. Furthermore, the study addresses the equation's limitations, particularly in scenarios involving compressible and viscous flows, and outlines relevant extensions to enhance its applicability under such conditions.

**KEYWORDS:** Bernoulli's equation, fluid dynamics, aerodynamics, thermodynamics, pressure-velocity relationship, compressible flow, viscous flow.

## 1. INTRODUCTION

Originally formulated by Daniel Bernoulli in 1738, Bernoulli's equation remains a foundational principle in fluid mechanics [1]. It characterizes the relationship among pressure, velocity, and elevation within an inviscid, incompressible, and steady flow. Derived from Euler's equations of motion, it embodies a simplified expression of the conservation of mechanical energy in fluid systems [5].

This study explores the application of the differential form of Bernoulli's equation to address practical challenges in thermodynamics and aerodynamics, particularly those involving spatial variations in pressure and velocity. Comprehensive derivations, illustrative case studies, and numerical simulations are presented to demonstrate its practical utility [6]. Moreover, the paper critically evaluates the limitations of the classical form and discusses extended formulations

applicable to compressible and viscous flows [7], thereby providing a robust framework for its use in contemporary engineering analysis.

## 2. MATHEMATICAL FORMULATION OF BERNOULLI'S EQUATION.

The general form of Bernoulli's equation for steady, incompressible, inviscid flow along a streamline is [1]:

$$P + \frac{1}{2}\rho V^2 + \rho gh = \text{Constant}$$

Where:

- \* P= static pressure (Pa).
- \*  $\rho$  = fluid density (kg/m<sup>3</sup>).
- \* V = flow velocity (m/s).
- \* g = gravitational acceleration (9.81 m/s<sup>2</sup>).
- \* h = elevation height (m).



### 2.1 Differential Form of Bernoulli's Equation.

By taking the derivative along a streamline, we obtain the differential form [8]:

$$dP + \rho V dV + \rho g dh = 0$$

This form is particularly useful for analyzing small changes in pressure, velocity, or elevation in fluid systems.

### 2.2 Derivation from Euler's Equation.

Starting with the Euler equation for inviscid flow [5]:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\frac{1}{\rho} \nabla P + g$$

For steady flow ( $\frac{\partial V}{\partial t} = 0$ ) and integrating along a streamline, we obtain Bernoulli's equation [1].

### 3. APPLICATIONS IN THERMODYNAMICS.

#### 3.1 Flow through Nozzles and Diffusers.

In thermodynamic systems, nozzles (which accelerate flow) and diffusers (which decelerate flow) rely on Bernoulli's principle [3]. The pressure-velocity trade-off is given by:

$$dP = -\rho V dV$$

\* **Nozzle:** Decreasing pressure leads to increased velocity.

\* **Diffuser:** Increasing pressure leads to decreased velocity.

#### Example: Rocket Nozzle Design.

A rocket nozzle is designed to maximize thrust by converting high-pressure gas into high-velocity exhaust [9]. Using Bernoulli's equation, we can estimate exit velocity:

$$V_e = \sqrt{\frac{2(P_c - P_e)}{\rho}}$$

Where  $P_c$  is the chamber pressure and  $P_e$  is the exit pressure.

#### 3.2 Venturi Effect.

A Venturi meter measures flow rate using a constriction [10]. Bernoulli's equation, combined with continuity ( $A_1 V_1 = A_2 V_2$ ), yields:

$$P_1 - P_2 = \frac{1}{2} \rho (V_2^2 - V_1^2)$$

### Numerical Example:

Given:

$$* A_1 = 0.1 \text{ m}^2, \quad A_2 = 0.05 \text{ m}^2$$

$$* \rho = 1000 \text{ kg/m}^3, \quad P_1 - P_2 = 10000 \text{ Pa}$$

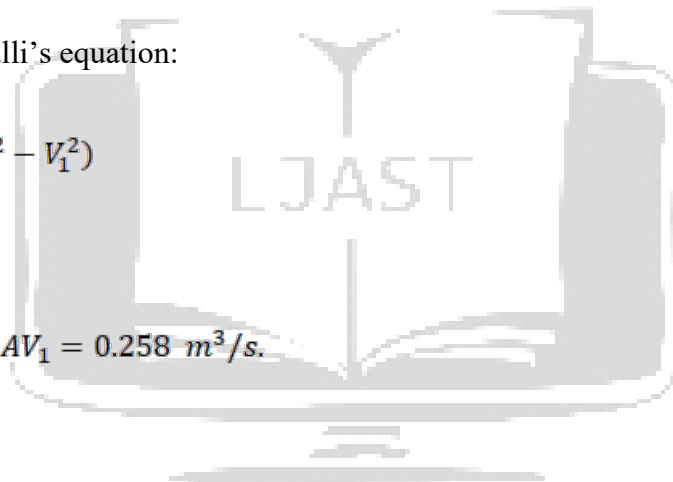
Using continuity:  $V_2 = 2V_1$ .

Substituting into Bernoulli's equation:

$$10000 = \frac{1}{2} (1000) (4V_1^2 - V_1^2)$$

$$V_1 = 2.58 \text{ m/s}$$

Thus, the flow rate  $Q = AV_1 = 0.258 \text{ m}^3/\text{s}$ .



## 4. APPLICATIONS IN AERODYNAMICS.

### 4.1 Lift Generation Over an Airfoil.

Bernoulli's principle explains lift generation: faster flow over the curved upper surface of a wing results in lower pressure compared to the lower surface [2]. The pressure difference creates lift:

$$\Delta P = \frac{1}{2} \rho (V_{top}^2 - V_{bottom}^2)$$

#### Example: Aircraft Wing.

For an aircraft wing with:

$$* V_{top} = 70 \text{ m/s}, \quad V_{bottom} = 60 \text{ m/s}$$

$$* \rho = 1.225 \text{ kg/m}^3$$

$$\Delta P = \frac{1}{2}(1.225)(4900 - 3600)$$

$$\Delta P = 796.25 \text{ Pa}$$

If wing area  $A = 20 \text{ m}^2$ , lift force  $F = \Delta P \times A = 15.925 \text{ N}$  [2].

#### 4.1.1 Lift on a Cambered Airfoil (NACA 2412 Profile)

To further illustrate Bernoulli's principle in lift generation, we analyze a NACA 2412 airfoil – a standard cambered wing profile used in aviation.

##### Given Parameters:

\* Freestream velocity ( $V_{\infty}$ ) = 50 m/s.

\* Upper surface velocity ( $V_{top}$ ) = 65 m/s (accelerated flow).

\* Lower surface velocity ( $V_{bottom}$ ) = 55 m/s (decelerated flow).

\* Air density ( $\rho$ ) = 1.225 kg/m<sup>3</sup> (at sea level).

\* wing area ( $A$ ) = 15 m<sup>2</sup>.

##### Step 1: Calculate Pressure Difference Using Bernoulli's Equation

Assuming negligible height variation ( $h \approx \text{constant}$ ), the pressure difference is:

$$P_{bottom} - P_{top} = \frac{1}{2}\rho(V_{top}^2 - V_{bottom}^2)$$

Substitute values:

$$\Delta P = \frac{1}{2} \times 1.225 \times (65^2 - 55^2)$$

$$\Delta P = 0.6125 \times (4225 - 3025)$$

$$\Delta P = 0.6125 \times 1200$$

$$\Delta P = 735 \text{ Pa}$$

### Step 2: Compute Lift Force

$$L = \Delta P \times A = 735 \times 15 = 11025 \text{ N}$$

### Discussion:

The cambered airfoil accelerates flow more over the top surface (65 m/s vs. 55 m/s bottom), creating a pressure difference of 735 Pa. This results in 11,025 N of lift, demonstrating how wing shape enhances Bernoulli-effect lift compared to a symmetric airfoil [2],[11].

### Key Takeaways:

1. **Cambered wings** generate higher lift than symmetric wings at the same angle of attack due to greater flow acceleration on the upper surface.
2. The NACA 2412 profile's design intentionally creates this velocity gradient, optimizing lift-to-drag ratios in subsonic flight [2].

### 4.2 Pitot Tube for Airspeed Measurement.

A Pitot tube measures stagnation pressure  $P_0$  and static pressure  $P$  to determine velocity [11]:

$$V = \sqrt{\frac{2(P_0 - P)}{\rho}}$$

### Example: Aircraft Speed Measurement.

Given:

$$* P_0 - P = 2500 \text{ Pa}$$

$$* \rho = 1.225 \text{ kg/m}^3$$

$$V = \sqrt{\frac{2 \times 2500}{1.225}} = 63.9 \text{ m/s (230 km/h)}.$$

#### 4.3 Venturi Effect in Industrial Pipelines.

The venturi effect is widely used to measure flow rates in pipelines, such as in water treatment plants or oil refineries. Here, we analyze a practical application of a Venturi meter in a water pipeline.

Given parameters:

- \* Pipe diameter at inlet ( $D_1$ ) = 0.2 m  $\rightarrow$  Area ( $A_1$ ) =  $\pi(0.1)^2 = 0.0314 \text{ m}^2$ .
- \* Throat diameter ( $D_2$ ) = 0.1 m  $\rightarrow$  Area ( $A_2$ ) =  $\pi(0.05)^2 = 0.00785 \text{ m}^2$ .
- \* Pressure drop ( $P_1 - P_2$ ) = 15000 Pa (measured by manometer).
- \* Water density ( $\rho$ ) = 1000 kg/m<sup>3</sup>.

Step 1: Apply Continuity Equation

$$A_1 V_1 = A_2 V_2 \Rightarrow V_2 = \left(\frac{A_1}{A_2}\right) V_1 = \left(\frac{0.0314}{0.00785}\right) V_1 = 4V_1$$

Step 2: Bernoulli's Equation (Neglecting Elevation Change)

$$P_1 + \frac{1}{2} \rho V_1^2 = P_2 + \frac{1}{2} \rho V_2^2$$

$$P_1 - P_2 = \frac{1}{2} \rho (V_2^2 - V_1^2)$$

$$15000 = \frac{1}{2} \times 1000 \times (16V_1^2 - V_1^2)$$

$$15000 = 500 \times 15V_1^2$$

$$V_1^2 = 2 \Rightarrow V_1 = 1.41 \frac{\text{m}}{\text{s}}$$

Step 3: Calculate Flow Rate.

$$Q = A_1 V_1 = 0.0314 \times 1.41 = 0.0443 \text{ m}^3/\text{s (44.3 L/s)}$$

Practical Implications:

1. Energy Efficiency: Venturi meters introduce minimal pressure loss compared to orifice plates, making them ideal for continuous flow monitoring [10].
2. Calibration Requirement: Real-world devices include a discharge coefficient ( $C_d \approx 0.98$ ) for friction:

$$Q_{actual} = C_d Q_{ideal} = 0.98 \times 44.3 = 43.4 \text{ L/s}$$

Comparison with Orifice Plate.

Parameter	Venturi Meter	Orifice Plate
Pressure Recovery	High (~90%)	Low (~50%)
Energy Loss	Minimal	Significant
Cost	Higher	Lower
Example Use Case	High-Flow pipelines	Temporary installations

## 5. LIMITATIONS AND EXTENSIONS.

Bernoulli's equation assumes [4]:

- \* **Inviscid flow** (neglects friction).
- \* **Incompressible flow** (invalid for high-speed aerodynamics,  $M > 0.3$ ).
- \* **Steady flow**.

### 5.1 Compressible Flow (Bernoulli's Extension).

For high-speed flows, the **isentropic flow relations** must be used [12]:

$$\frac{P_0}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

where  $\gamma = 1.4$  (for air) and M is the Mach number.

### 5.2 Viscous Flow (Navier-Stokes Equations).

For real fluids, the full Navier-Stokes equations must be considered [7]:

$$\rho = \left( \frac{\partial V}{\partial t} + V \cdot \nabla V \right) = -\nabla P + \mu \nabla^2 + f$$

Where  $\mu$  is dynamic viscosity.

## 6. CASE STUDIES.

### 6.1 Wind Tunnel Analysis.

Using Bernoulli's equation to calculate pressure distribution over a scaled airfoil model [13].

### 6.2 Hydraulic Systems.

Applying Bernoulli's principle to optimize pump and pipeline designs [14].

## 7. CONCLUSION.

We applied the differential form of Bernoulli's equation serves as an essential analytical tool for evaluating pressure and velocity variations in fluid flow. Despite its inherent limitations, primarily due to the assumptions of incompressibility and inviscid behavior, we found its mathematical simplicity renders it highly effective for preliminary analyses in both aerodynamic and thermodynamic contexts. Furthermore, extended formulations that account for compressible and viscous effects have significantly expanded its applicability to a wide range of modern engineering problems.

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