

STUDY EXACT SOLUTIONS AND ANALYTICAL OF LINEAR VIE'S USING NON-POLYNOMIAL SPLINE FUNCTIONS

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Abstract

In this paper, both linear and quadratic of non-polynomial spline functions have been applied to find the numerical solution of VIE's. New algorithms have been proposed for the first time which is essential in this work.

In section (2) linear VIE's of the second kind have been solved using linear and quadratic non-polynomial spline functions in subsection (2.1) and (2.2) respectively. In section (3), the solution of linear VIE's of the first kind with $k(x, x) \neq 0$ will be introduced. In section (4), we used linear and quadratic non-polynomial spline functions to solve VIE's with weak singular kernel in subsection (4.1) and (4.2) respectively. In section (5), numerical examples are given for illustrations and comparison between the method has been made and polynomial spline function and other known methods. And finally, in section (6), including a discussion for this.

Algorithm developed for solving those equations using MATLAB programming.

Keywords: exact solutions, numerical solutions, Non-polynomial, Spline function, analytical solutions, Volterra integral equation, weakly singular kernel.

-1- Introduction.

In 1997, Diogo and T-Lima presented an inductive solution method for finding the numerical solution to Volterra integral equation with weak singular kernel. In 2012, Hossinpour solved integral differential equations using non-polynomial spline function.

Integral equations find special applicability within scientific and mathematical disciplines. To check the numerical method, it is applied to solve different test problems with known exact solutions and the numerical solutions obtained confirm the validity of the numerical method. Many types of equations do not have an analytical solution. Therefore, these problems should be solved by using numerical techniques. In numerical methods, computer codes and more powerful processors are required to achieve accurate results. Volterra integral equation arises in many scientific applications such as the population dynamic, spread of epidemics [8].

-1.1- Definition. [4]

The **normalized B-spline** $B_{i,k+1}$ of degree k relative to the distinct nodes $x_i, \dots, x_{i+k+1}$ is defined as:

$$B_{i,k+1} = (x_{i+k+1} - x_i)g(x_i, \dots, x_{i+k+1}) \quad (1)$$

$$\text{where } g(t) = (t - x)^k = \begin{cases} (t - x)^k & \text{if } x \leq t \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

By Newton form of the interpolating polynomial we have, ([4]).

$$f(x_i, \dots, x_n) = \sum_{i=0}^n \frac{f(x_i)}{W'_{n+1}(x_i)} \quad (3)$$

Where $W_{n+1} = \prod_0^n (x - x_i)$

-1.2- Linear Non-Polynomial Spline Function.

The form of the linear non-polynomial spline function is:

$$p_i(t) = a_i \cos k(t - t_i) + b_i \sin k(t - t_i) + c_i(t - t_i) + d_i, i = 0, 1, \dots, n \quad (4)$$

where are constants to be determined. In order to obtain the value of, we differentiate equation (4) three times with respect to t , then we get:

$$\left. \begin{aligned} p'_i(t) &= -ka_i \sin k(t - t_i) + kb_i \cos k(t - t_i) + c_i \\ p''_i(t) &= -k^2 a_i \cos k(t - t_i) - k^2 b_i \sin k(t - t_i) \\ p'''_i(t) &= k^3 a_i \sin k(t - t_i) - k^3 b_i \cos k(t - t_i) \end{aligned} \right\} \quad (5)$$

Hence replace t by in the relation (4) and (5) yields:

$$p_i(t_i) = a_i + d_i$$

$$p'_i(t_i) = kb_i + c_i$$

$$p''_i(t_i) = -k^2 a_i$$

$$p'''_i(t_i) = -k^3 b_i$$

From the above equations, the values of are obtained as follows:

$$a_i = \frac{-1}{k^2} p''_i(t_i) \quad (6)$$

$$b_i = \frac{-1}{k^3} p'''_i(t_i) \quad (7)$$

$$c_i = p'_i(t_i) + kb_i \quad (8)$$

$$d_i = p_i(t_i) + a_i \quad (9)$$

for $i = 0, 1, \dots, n$

-1.3- Quadratic Non -Polynomial Spline Function.

The form of the quadratic non-polynomial spline function is:

$$Q_i(t) = a_i \cos k(t - t_i) + b_i \sin k(t - t_i) + c_i(t - t_i) + d_i(t - t_i)^2 + e_i \quad (10)$$

, $i = 0, 1, \dots, n$ where a_i, b_i, c_i, d_i and e_i are constants to be determined. In order to obtain the values of a_i, b_i, c_i, d_i and e_i , we differentiate equation (10) four times with respect to t , and then we get the following equations:

$$\left. \begin{aligned} Q_i'(t) &= -ka_i \sin k(t - t_i) + kb_i \cos k(t - t_i) + c_i + 2d_i(t - t_i) \\ Q_i''(t) &= -k^2 a_i \cos k(t - t_i) - k^2 b_i \sin k(t - t_i) + 2d_i \\ Q_i'''(t) &= k^3 a_i \sin k(t - t_i) - k^3 b_i \cos k(t - t_i) \\ Q_i''''(t) &= k^4 a_i \cos k(t - t_i) + k^4 b_i \sin k(t - t_i) \end{aligned} \right\} \quad (11)$$

Hence replace t by t_i in the relation (10) and (11) yields:

$$Q_i(t_i) = a_i + e_i$$

$$Q_i'(t_i) = kb_i + c_i$$

$$Q_i''(t_i) = -k^2 a_i + 2d_i$$

$$Q_i'''(t_i) = -k^3 b_i$$

$$Q_i''''(t_i) = k^4 a_i$$

We obtain the values of from the above relations as follows:

$$a_i = \frac{1}{k^4} Q_i''''(t_i) \quad (12)$$

$$b_i = \frac{-1}{k^3} Q_i'''(t_i) \quad (13)$$

$$c_i = Q_i'(t_i) - kb_i \quad (14)$$

$$d_i = \frac{1}{2} [Q_i''(t_i) + k^2 a_i] \quad (15)$$

$$e_i = Q_i(t_i) - a_i \quad (16)$$

for $i = 0, 1, \dots, n$

-2- Solution of Linear VIE's of the Second Kind.

In this section, the linear and quadratic non-polynomial spline functions have been used to find the numerical solution of linear VIE's of the second kind, which has the form:

$$u(x) = f(x) + \int_a^x k(x, t)u(t)dt, \quad x \in [a, b] \quad (17)$$

Where $k(x, t)$ and $f(x)$ are known functions and continuous in $C^4[a, b]$ but $u(x)$ is unknown function. To solve the equation (17), we need to differentiate equation (17) four times with respect to x , by using Libenze formula we realize:

$$u'(x) = f'(x) + \int_a^x \frac{\partial k(x, t)}{\partial x} u(t)dt + k(x, x)u(x) \quad (18)$$

$$\begin{aligned} u''(x) &= f''(x) + \int_a^x \frac{\partial^2 k(x, t)}{\partial x^2} u(t)dt + \left(\frac{\partial k(x, t)}{\partial x}\right)_{t=x} u(x) + \frac{dk(x, x)}{dx} u(x) \\ &\quad + k(x, x)u'(x) \end{aligned} \quad (19)$$

$$u'''(x) = f'''(x) + \int_a^x \frac{\partial^3 k(x, t)}{\partial x} u(t) dt + \left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} u(x) + \frac{d}{dx} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} u(x) \\ + \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} u'(x) + \frac{d^2 k(x, x)}{dx^2} u(x) + 2 \frac{dk(x, x)}{dx} u'(x) + k(x, x) u''(x) \quad (20)$$

$$u^{(4)}(x) = f^{(4)}(x) + \int_a^x \frac{\partial^4 k(x, t)}{\partial x^4} u(t) dt + \left(\frac{\partial^3 k(x, t)}{\partial x^3} \right)_{t=x} u(x) + \frac{d}{dx} \left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} u(x) \\ + \left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} u'(x) + \frac{d^2}{dx^2} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} u(x) + \frac{d}{dx} \left(\frac{\partial k(x, t)}{\partial x^2} \right)_{t=x} u'(x) \\ + \frac{d}{dx} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} u'(x) + \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} u''(x) + \frac{d^3 k(x, x)}{dx^3} u(x) \\ + 3 \frac{d^2 k(x, x)}{dx^2} u'(x) + 3 \frac{dk(x, x)}{dx} u''(x) + k(x, x) u'''(x) \quad (21)$$

To complete our procedure for solving VIE's. we substitute $x = a$ in equations (17) - (21), then we get:

$$u_0 = u(a) = f(a) \quad (22)$$

$$u'_0 = u'(a) = f'(a) + k(a, a)u(a) \quad (23)$$

$$u''_0 = u''(a) = f''(a) + \left[\left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u(a) + \left(\frac{\partial k(x, t)}{\partial x} \right)_{x=a} \frac{dk(x, x)}{dx} u(a) \\ + k(a, a)u'(a) \quad (24)$$

$$u'''_0 = u'''(a) = f'''(a) + \left[\left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} \right]_{x=a} u(a) + \left[\frac{d}{dx} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u(a) \\ + \left[\left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u'(a) + \left[\frac{d^2 k(x, x)}{dx^2} \right]_{x=a} u(a) + 2 \left[\frac{dk(x, x)}{dx} \right]_{x=a} u'(a) \\ + k(a, a)u''(a) \quad (25)$$

$$u^{(4)}_0 = u^{(4)}(a) = f^{(4)}(a) + \left[\left(\frac{\partial^3 k(x, t)}{\partial x^3} \right)_{t=x} \right]_{x=a} u(a) + \left[\frac{d}{dx} \left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} \right]_{x=a} u(a) \\ + \left(\frac{\partial^2 k(x, t)}{\partial x^2} \right)_{t=x} u'(a) + \left[\frac{d^2}{dx^2} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u(a) \\ + 2 \left[\frac{d}{dx} \left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u'(a) + \left[\left(\frac{\partial k(x, t)}{\partial x} \right)_{t=x} \right]_{x=a} u''(a) + \left[\frac{d^3 k(x, x)}{dx^3} \right]_{x=a} u(a) \\ + 3 \left[\frac{d^2 k(x, x)}{dx^2} \right]_{x=a} u'(a) + 3 \left[\frac{dk(x, x)}{dx} \right]_{x=a} u''(a) \\ + k(a, a)u'''(a) \quad (26)$$

Now, we try to solve equation (17) using linear and quadratic nonpolynomial spline functions

-2.1- Using Linear Non-Polynomial Spline Function.

We approximate the solution of linear VIE's of the second kind (17) by using linear non-polynomial spline function (3). We introduce a method of solution in algorithm (VIE2NPS1):

The Algorithm (VIE2NPS1).

Step 1: Set $h = \frac{b-a}{n}$, $t_i = t_0 + ih, i = 0, \dots, n$ (where $t_0 = a, t_n = b$) and $u_0 = f(a)$.

Step 2: Evaluate a_0, b_0, c_0 and d_0 and by substituting (22) -(25) in equations (6) - (9).

Step 3: Calculate $P_0(t)$ using step2 and equation (3).

Step 4: Approximate $u_1 = P_0(t_1)$.

Step 5: For $i = 1$ to $n - 1$ do the following steps:

Step6: Evaluate a_i, b_i, c_i and d_i by using equations (6) -(9) and replacing $u(t_i), u'(t_i), u''(t_i)$ and $u'''(t_i)$ by $P_i(t_i), P_i'(t_i), P_i''(t_i)$ and $P_i'''(t_i)$.

Step 7: Calculate $P_i(t)$ using step 6 and equation (3).

Step 8: Approximate $u_{i+1} = P_i(t_{i+1})$.

-2.2- Using Quadratic Non-Polynomial Spline Function:

In order to approximate the solution of linear VIE's of second kind (17) by using quadratic non-polynomial spline function (10). We present a method of solution in algorithm (VIE2NPS2):

The Algorithm (VIE2NPS2):

Step 1: Set $h = \frac{b-a}{n}$, $t_i = t_0 + ih, i = 0, \dots, n$ (where $t_0 = a, t_n = b$) and $u_0 = f(a)$.

Step 2: Evaluate a_0, b_0, c_0, d_0 and e_0 and by substituting (22) -(26) in equations (12) - (16).

Step 3: Calculate $P_0(t)$ using step2 and equation (10).

Step 4: Approximate $u_1 = P_0(t_1)$.

Step 5: For $i = 1$ to $n - 1$ do the following steps:

Step6: Evaluate a_i, b_i, c_i, d_i and e_i by using equations (12) -(16) and replacing $u(t_i), u'(t_i), u''(t_i), u'''(t_i)$ and $u^{(4)}(t_i)$ by $P_i(t_i), P_i'(t_i), P_i''(t_i), P_i'''(t_i)$ and $P_i^{(4)}(t_i)$.

Step 7: Calculate $P_i(t)$ using step 6 and equation (10).

Step 8: Approximate $u_{i+1} = P_i(t_{i+1})$.

-3- Solution of VIE's of the FIRST Kind.

In this section, we introduce the solution of VIE's of the first kind which has a form:

$$g(x) = \int_a^x r(x, t)u(t)dt, \quad x \in [a, b] \quad (27)$$

where $r(x, t)$ and $g(x)$ are known functions and continuous in $C^4[a, b]$, but $u(x)$ is unknown function. When $r(x, t) \neq 0$, we differential equations (27) one time with respect to x . Therefore, we get conversion to the second kind, i.e.;

$$u(x) = \frac{1}{r(x, x)} \left[g'(x) - \int_a^x \frac{\partial r(x, t)}{\partial x} u(t)dt \right] \quad (28)$$

then we use the non-polynomial spline function and algorithms (VIE2NPS1) and (VIE2NPS2) to solve (28).

$$\text{where } k(x, t) = \frac{1}{r(x, x)} * \frac{\partial r(x, t)}{\partial x} \text{ and } f(x) = \frac{1}{r(x, x)} g'(x)$$

-4- Linear VIE's of the Second Kind with Weakly Singular kernel.

In this section, the linear and quadratic non-polynomial spline functions will be used to compute the numerical solution of linear VIE's of second kind with weakly singular kernel, which is:

$$u(x) - \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t) dt = f(x), \quad x \in [0, T] \quad (29)$$

where $0 < \mu < 1$ and f is known function. There is a singularity at $x = 0$ and $t = 0$ for any positive value of x . In order to solve (29), we multiply both sides of (29) by x^μ yields to:

$$x^\mu u(x) - \int_0^x t^{\mu-1} u(t) dt = f(x) x^\mu \quad (30)$$

Hence differentiation (30) with respect to x , we get:

$$x^\mu u'(x) + \mu x^{\mu-1} u(x) - \frac{1}{x^{1-\mu}} u(x) = \mu x^{\mu-1} f(x) + f'(x) x^\mu \quad (31)$$

And multiplication both sides of (31) by $x^{1-\mu}$ yields,

$$x u'(x) + (\mu - 1) u(x) = \mu f(x) + x f'(x)$$

Remark 1: Note that $\lim_{x \rightarrow 0} \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t) dt = \frac{u(0)}{\mu}$. Therefore, if $u(0) \neq 0$.

We have $u(0) \neq f(0)$, more precisely $u_0 = \frac{\mu}{\mu-1} f(0)$.

Hence equation (29) may be converted into the following form [2]:

$$x u'(x) + (\mu - 1) u(x) = \mu f(x) + x f'(x) \quad (32)$$

With

$$u_0 = \frac{\mu}{\mu-1} f(0) \quad (33)$$

Hence differentiate equation (32) four times with respect to x , we get:

$$\left. \begin{aligned} x u''(x) + \mu u'(x) &= (\mu + 1) f'(x) + x f''(x) \\ x u'''(x) + (\mu + 1) u''(x) &= (\mu + 2) f''(x) + x f'''(x) \\ x u^{(4)}(x) + (\mu + 2) u'''(x) &= (\mu + 3) f'''(x) + x f^{(4)}(x) \\ x u^{(5)}(x) + (\mu + 3) u^{(4)}(x) &= (\mu + 4) f^{(4)}(x) + x f^{(5)}(x) \end{aligned} \right\} \quad (34)$$

Hence replace x by a in the relation, yields:

$$\left. \begin{aligned} u'_0 &= \frac{\mu+1}{\mu} f'(a) \\ u''_0 &= \frac{\mu+2}{\mu+1} f''(a) \\ u'''_0 &= \frac{\mu+3}{\mu+2} f'''(a) \\ u^{(4)}_0 &= \frac{\mu+4}{\mu+3} f^{(4)}(a) \end{aligned} \right\} \quad (35)$$

-4.1- Using Linear Non-Polynomial Spline Function.

In order to approximate the solution of linear VIE's of the second kind with weakly singular kernel (29) by using linear non-polynomial spline function (3). We present a method of solution in algorithm (VIE2WSKNPS1):

The Algorithm: (VIE2WSKNPS1):

Step 1: Set $h = \frac{b-a}{n}$, $t_i = t_0 + ih, i = 0, 1, \dots, n$ (where $t_0 = a, t_n = b$) and $u_0 = \frac{\mu}{\mu-1} f(a)$.

Step 2: Evaluate a_0, b_0, c_0 and d_0 and by substituting (33) -(35) in equations (6) - (9).

Step 3: Calculate $P_0(t)$ using step2 and equation (3).

Step 4: Approximate $u_1 = P_0(t_1)$.

Step 5: For $i = 1$ to $n - 1$ do the following steps:

Step6: Evaluate a_i, b_i, c_i and d_i by using equations (6) -(9) and replacing $u(t)$ and its derivatives by $P_i(t)$ and its derivative's.

Step 7: Calculate $P_i(t)$ using step 6 and equation (3).

Step 8: Approximate $u_{i+1} = P_i(t_{i+1})$.

-4.2- Using Quadratic Non-Polynomial Spline Function.

We approximate the solutions of linear VIE's of the second kind with weakly singular kernel by using quadratic non-polynomial spline function (10). In the following algorithm (VIE2WSKNPS2):

The Algorithm (VIE2WSKNPS2):

Step 1: Set $h = \frac{b-a}{n}$, $t_i = t_0 + ih, i = 0, 1, \dots, n$ (where $t_0 = a, t_n = b$) and $u_0 = \frac{\mu}{\mu-1} f(a)$.

Step 2: Evaluate a_0, b_0, c_0, d_0 and e_0 and by substituting (33) -(35) in equations (12) - (16).

Step 3: Calculate $P_0(t)$ using step2 and equation (10).

Step 4: Approximate $u_1 = P_0(t_1)$.

Step 5: For $i = 1$ to $n - 1$ do the following steps:

Step6: Evaluate a_i, b_i, c_i, d_i and e_i substituting in equations (12) -(16) and replacing $u(t)$ and its derivatives by $P_i(t)$ and its derivative's.

$u(t_i), u'(t_i), u''(t_i), u'''(t_i)$ and $u^{(4)}(t_i)$ by $P_i(t_i), P_i'(t_i), P_i''(t_i), P_i'''(t_i)$ and $P_i^{(4)}(t_i)$.

Step 7: Calculate $P_i(t)$ using step 6 and equation (10).

Step 8: Approximate $u_{i+1} = P_i(t_{i+1})$.

-5- Numerical Examples.

Example (1): Consider the VIE of the second kind [5].

$$\phi(x) = x + \int_0^x (t-x)\phi(t)dt \quad 0 \leq x \leq 1$$

With exact solution $\phi(x) = \sin x$.

Table (5.1) presents a comparison between the exact and numerical solution using linear and quadratic nonpolynomial spline functions, where $P_i(x)$ denotes the approximate solution using non-polynomial spline function, with $h=0.1$.

Table 5.1: Exact and numerical solution of test example (1).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	0	0	0
0.1	0.099833416646828	0.099833416646828	0.099833416646828
0.2	0.198669330795061	0.198669330795061	0.198669330795061
0.3	0.295520206661340	0.295520206661340	0.295520206661340
0.4	0.389418342308651	0.389418342308650	0.389418342308650
0.5	0.479425538604203	0.479425538604203	0.479425538604203
0.6	0.564642473395035	0.564642473395035	0.564642473395035
0.7	0.644217687237691	0.644217687237691	0.644217687237691
0.8	0.717356090899523	0.717356090899523	0.717356090899523
0.9	0.783326909627483	0.783326909627483	0.783326909627483
1	0.841470984807897	0.841470984807896	0.841470984807896

Table (5.2) present a comparison between the error in our methods and other method in [5], where Error = |exact value – numerical value| and $\|err\|_{\infty} = \max |Error|$.

Table 5.2: comparison between the error with reference [5].

x	Error in linear	Error in quadratic	Error obtains in [5]
0	0	0	0
0.1	0	0	2.0508063e-012
0.2	0	0	8.3558996e-013
0.3	5.5511151231258e-17	5.5511151231258e-17	2.0756558e-013
0.4	1.11022302462516e-16	1.11022302462516e-16	2.4960310e-013
0.5	1.11022302462516e-16	1.11022302462516e-16	3.6565473e-013
0.6	1.11022302462516e-16	1.11022302462516e-16	1.5317015e-013
0.7	1.11022302462516e-16	1.11022302462516e-16	1.1908461e-013
0.8	2.2244604925031e-16	2.2244604925031e-16	2.5211375e-013
0.9	3.33066907387547e-16	3.33066907387547e-16	2.8431391e-013
1	4.44089209850063e-16	4.44089209850063e-16	8.8095244e-013
$\ err\ _{\infty}$	4.44089209850063e-16	4.44089209850063e-16	2.050806e-012

Table (5.3) present a comparison between error obtain using linear and quadratic non-polynomial spline functions and polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)), with $h=0.1$.

Table 5.3: Comparison between error obtain using polynomial and nonpolynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	0.000166583353	0.000166583353
0.2	0	0	0.098669330795	0.098669330795
0.3	5.55111512312e-17	5.55111512312e-17	0.095520206661	0.095520206661
0.4	1.110223024625e-16	1.110223024625e-16	0.189418342308	0.189418342308
0.5	1.110223024625e-16	1.110223024625e-16	0.179425538604	0.179425538604
0.6	1.110223024625e-16	1.110223024625e-16	0.264642473395	0.264642473395
0.7	1.110223024625e-16	1.110223024625e-16	0.244217687237	0.244217687237
0.8	2.224460492503e-16	2.224460492503e-16	0.317356090899	0.317356090899
0.9	3.330669073875e-16	3.330669073875e-16	0.283326909627	0.283326909627
1	4.440892098500e-16	4.440892098500e-16	0.341470984807	0.341470984807
$\ err\ _{\infty}$	4.440892098500e-16	4.440892098500e-16	0.341470984807	0.341470984807

Example (2): Consider the VIE of the second kind [7].

$$y(x) = 1 + \int_0^x (t-x)y(t)dt \quad 0 \leq x \leq 1$$

With exact solution $y(x) = \cos x$.

Tables (5.4) present a comparison between the exact and numerical solution of linear and quadratic nonpolynomial spline functions where $P_i(x)$ denote the approximate solution using non-polynomial spline function, with $h=0.1$.

Table 5.4: Exact and numerical solution of test example (2).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	1.0000000000000000	1.0000000000000000	1.0000000000000000
0.1	0.995004165278026	0.995004165278026	0.995004165278026
0.2	0.980066577841242	0.980066577841242	0.980066577841242
0.3	0.955336489125606	0.955336489125606	0.955336489125606
0.4	0.921060994002885	0.921060994002885	0.921060994002885
0.5	0.877582561890373	0.877582561890373	0.877582561890373
0.6	0.825335614909678	0.825335614909678	0.825335614909678
0.7	0.764842187284489	0.764842187284488	0.764842187284488
0.8	0.696706709347165	0.696706709347165	0.696706709347165
0.9	0.621609968270664	0.621609968270664	0.621609968270664
1	0.540302305868140	0.540302305868139	0.540302305868139

Table (5.5) present a comparison between the error in our methods and other method in [7], where error = |exact value – numerical value| and $\|err\|_{\infty} = \max |Error|$.

Table 5.5: comparison between the error with reference [7].

x	Error in linear	Error in quadratic	Error obtains in [7]
0	0	0	-
0.1	0	0	-
0.2	0.1110223024e-16	0.1110223024e-16	0
0.3	0.1110223024e-16	0.1110223024e-16	-
0.4	0.2220446049e-16	0.2220446049e-16	0
0.5	0.2220446049e-16	0.2220446049e-16	-
0.6	0.2220446049e-16	0.2220446049e-16	8.993e-015
0.7	0.3330669073e-16	0.3330669073e-16	-
0.8	0.3330669073e-16	0.3330669073e-16	5.031e-013
0.9	0.3330669073e-16	0.3330669073e-16	-

1	444089209850e-16	444089209850e-16	1.142e-011
$\ err\ _{\infty}$	444089209850e-16	444089209850e-16	1.142e-011

Table (5.6) present a comparison between error obtain using linear and quadratic non-polynomial spline functions and polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)), with h=0.1.

Table 5.6: Comparison between error obtain using polynomial and nonpolynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	0.004995834721	0.000004165278
0.2	0.1110223024e-16	0.1110223024e-16	0.019933422158	0.004933422158
0.3	0.1110223024e-16	0.1110223024e-16	0.044663510874	0.014663510874
0.4	0.2220446049e-16	0.2220446049e-16	0.078939005997	0.028939005997
0.5	0.2220446049e-16	0.2220446049e-16	0.122417438109	0.047417438109
0.6	0.2220446049e-16	0.2220446049e-16	0.174664385093	0.069664385090
0.7	0.3330669073e-16	0.3330669073e-16	0.235157812715	0.095157812715
0.8	0.3330669073e-16	0.3330669073e-16	0.303293290652	0.123293290652
0.9	0.3330669073e-16	0.3330669073e-16	0.378390031729	0.153390031729
1	444089209850e-16	444089209850e-16	0.459697694131	0.184697694131
$\ err\ _{\infty}$	444089209850e-16	444089209850e-16	0.459697694131	0.184697694131

Example (3): Consider the VIE of the second kind [6].

$$u(x) = 1 - x + \frac{x^2}{2} + \int_0^x (t - x)u(t)dt \quad 0 \leq x \leq 1$$

With exact solution $u(x) = (1 - \sin x)$.

Tables (5.7) present a comparison between the exact and numerical solution using linear and quadratic non-polynomial spline functions, where $P_i(x)$ denote the approximate solution using non-polynomial spline functions, with h=0.1

Table 5.7: Exact and numerical solution of test example (3).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	1.0000000000000000	1.0000000000000000	1.0000000000000000
0.1	0.900166583353172	0.900166583353172	0.900166583353172
0.2	0.801330669204939	0.801330669204939	0.801330669204939

0.3	0.704479793338660	0.704479793338660	0.704479793338660
0.4	0.610581657691349	0.610581657691350	0.610581657691350
0.5	0.520574461395797	0.520574461395797	0.520574461395797
0.6	0.435357526604965	0.435357526604965	0.435357526604965
0.7	0.355782312762309	0.355782312762309	0.355782312762309
0.8	0.282643909100477	0.282643909100477	0.282643909100477
0.9	0.216673090372517	0.216673090372517	0.216673090372517
1	0.158529015192104	0.158529015192104	0.158529015192104

Table (5.8) present a comparison between the error in our methods and other method in [6], where error = |exact value – numerical value| and $\|err\|_{\infty} = \max|Error|$.

Table 5.8: comparison between the error with reference [6].

x	Error in linear	Error in quadratic	Error obtains in [6]
0	0	0	-
0.1	0	0	-
0.2	0	0	-
0.3	0	0	-
0.4	2.220446049250310e-16	2.220446049250310e-16	-
0.5	1.110223024625160e-16	1.110223024625160e-16	-
0.6	1.110223024625160e-16	1.110223024625160e-16	-
0.7	1.110223024625160e-16	1.110223024625160e-16	-
0.8	2.220446049250310e-16	2.220446049250310e-16	-
0.9	3.330669073875471e-16	3.330669073875471e-16	-
1	4.440892098500626e-16	4.440892098500626e-16	-
$\ err\ _{\infty}$	4.440892098500626e-16	4.440892098500626e-16	3.6208210e-04

Table (5.9) present a comparison between error obtain using nonpolynomial spline function including linear and quadratic polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)) with $h=0.1$.

Table 5.9: Comparison between error obtain using polynomial and nonpolynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	0.0001665833531	0.0001665833531
0.2	0	0	0.0986693307950	0.0986693307950
0.3	0	0	0.0955202066613	0.0955202066613

0.4	2.2204460492e-16	2.2204460492e-16	0.1894183423086	0.1894183423086
0.5	1.1102230246e-16	1.1102230246e-16	0.1794255386042	0.1794255386042
0.6	1.1102230246e-16	1.1102230246e-16	0.2646424733950	2646424733950
0.7	1.1102230246e-16	1.1102230246e-16	0.2442176872376	0.2442176872376
0.8	2.2204460492e-16	2.2204460492e-16	0.3173560908995	0.3173560908995
0.9	3.3306690738e-16	3.3306690738e-16	0.2833269096274	0.2833269096274
1	4.4408920985e-16	4.4408920985e-16	0.3414709848078	0.3414709848078
$\ err\ _{\infty}$	4.4408920985e-16	4.4408920985e-16	0.3414709848078	0.3414709848078

Example (4): Consider the VIE of the first kind [3].

$$\int_0^x \cos(x-t) y(t) dt = \sin x \quad 0 \leq x \leq 1$$

And Exact solution $y(x) = 1$, To solve this equation we differentiate the equation above one time with respect to x , to obtain.

$$y(x) = \cos x + \int_0^x \sin(x-t) y(t) dt$$

Which is second kind VIE's, with $f(x) = \cos x$ and $k(x, t) = \sin(x-t)$.

Tables (5.10) present a comparison between the exact and numerical solution using linear and quadratic non-polynomial spline functions, where $P_i(x)$ denote the approximate solution using non-polynomial spline functions, with $h=0.1$.

Table 5.10: Exact and numerical solution of test example (4).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	1	1	1
0.1	1	1	1
0.2	1	1	1
0.3	1	1	1
0.4	1	1	1
0.5	1	1	1
0.6	1	1	1
0.7	1	1	1
0.8	1	1	1
0.9	1	1	1
1	1	1	1

Table (5.11) present a comparison between the error in our methods and other method in [3], where $\text{error} = |\text{exact value} - \text{numerical value}|$ and $\|err\|_{\infty} = \max|\text{Error}|$.

Table 5.11: comparison between the error with reference [3].

x	Error in linear	Error in quadratic	Error obtains in [3]
0	0	0	-
0.1	0	0	1.00166
0.2	0	0	-
0.3	0	0	1.00166
0.4	0	0	-
0.5	0	0	1.00166
0.6	0	0	-
0.7	0	0	-
0.8	0	0	-
0.9	0	0	-
1	0	0	-
$\ err\ _{\infty}$	0	0	-

Table (5.12) present a comparison between error obtain using linear and quadratic non-polynomial and polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)) with $h=0.1$.

Table 5.12: Comparison between error obtain using polynomial and non-polynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0	0	0	0
0.4	0	0	0	0
0.5	0	0	0	0
0.6	0	0	0	0
0.7	0	0	0	0
0.8	0	0	0	0
0.9	0	0	0	0
1	0	0	0	0
$\ err\ _{\infty}$	0	0	0	0

Example (5): Consider the VIE of second kind with Weakly Singular Kernel [2].

$$u(x) - \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t) dt = f(x) \quad 0 \leq x \leq 1$$

Where $f(x) = x + 1$ and $\mu = 0.5$, with $u(x) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} x$.

Tables (5.13) present a comparison between the exact and numerical solution using linear and quadratic non-polynomial spline functions, where $P_i(x)$ denote the approximate solution non-polynomial spline functions, with $h=0.1$.

Table 5.13: Exact and numerical solution of test example (5).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	-1.0000000000000000	-1.0000000000000000	-1.0000000000000000
0.1	-0.7000000000000000	-0.7000000000000000	-0.7000000000000000
0.2	-0.4000000000000000	-0.4000000000000000	-0.4000000000000000
0.3	-0.1000000000000000	-0.1000000000000000	-0.1000000000000000
0.4	0.2000000000000000	0.2000000000000000	0.2000000000000000
0.5	0.5000000000000000	0.5000000000000000	0.5000000000000000
0.6	0.8000000000000000	0.8000000000000000	0.8000000000000000
0.7	1.1000000000000000	1.1000000000000000	1.1000000000000000
0.8	1.4000000000000000	1.4000000000000000	1.4000000000000000
0.9	1.7000000000000000	1.7000000000000000	1.7000000000000000
1	2.0000000000000000	2.0000000000000000	2.0000000000000000

Table (5.14) present a comparison between the error in our methods and other method in [2] where $\text{error} = |\text{exact value} - \text{numerical value}|$ and $\|err\|_\infty = \max|\text{Error}|$.

Table 3.26: comparison between the error with reference [2].

x	Error in linear	Error in quadratic	Error obtains in [2]
0.08	2.220446049250300e-16	2.220446049250300e-16	2.1e-0.4
0.16	4.440892098500600e-16	4.440892098500600e-16	5.3e-0.4
0.24	5.551115123125801e-16	5.551115123125801e-16	1.3e-0.3
0.48	5.551115123125801e-16	5.551115123125801e-16	1.3e-0.3
0.64	9.992007221626401e-16	9.992007221626401e-16	8.0e-0.4
0.80	1.332267629550190e-15	1.332267629550190e-15	6.1e-0.4
0.96	2.220446049250310e-15	2.220446049250310e-15	5.1e-0.4
1	2.220446049250310e-15	2.220446049250310e-15	4.9e-0.4
$\ err\ _\infty$	2.220446049250313e-15	2.220446049250313e-15	-

Table (5.15) present a comparison between error obtain using nonpolynomial spline function including linear and polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)), with $h=0.1$.

Table 5.15: Comparison between error obtain using polynomial and nonpolynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0	0	0	0
0.4	0	0	0	0
0.5	0	0	0	0
0.6	0	0	0	0
0.7	0	0	0	0
0.8	0	0	0	0
0.9	0	0	0	0
1	0	0	0	0
$\ err\ _{\infty}$	0	0	0	0

Example (6): Consider the VIE of second with Weakly Singular Kernel [1].

$$u(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t) dt = f(x) \quad 0 \leq x \leq 1$$

Where $f(x) = x^2 + x + 1$ and $\mu = 0.5$, with $u(x) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} x + \frac{\mu+2}{\mu+1} x^2$.

Tables (5.16) present a comparison between the exact and numerical solution of non-polynomial spline function including linear and quadratic, where $P_i(x)$ denote the approximate solution use non-polynomial spline functions, with $h=0.1$.

Table 5.16: Exact and numerical solution of test example (6).

x	Exact solution	$P_i(x)$	
		linear	quadratic
0	-1.0000000000000000	-1.0000000000000000	-1.0000000000000000
0.1	-0.6833333333333333	-0.683347217593419	-0.6833333333333333
0.2	-0.3333333333333333	-0.333555259470805	-0.3333333333333333
0.3	0.0500000000000000	0.048878369581314	0.0500000000000000
0.4	0.4666666666666667	0.463130019990385	0.4666666666666667
0.5	0.9166666666666667	0.908058127032092	0.9166666666666667

0.6	1.4000000000000000	1.382214616967740	1.4000000000000000
0.7	1.9166666666666666	1.883859375718374	1.9166666666666667
0.8	2.4666666666666667	2.410977635509450	2.4666666666666666
0.9	3.0500000000000001	2.961300105764454	3.0500000000000000
1	3.6666666666666667	3.532325647106203	3.6666666666666667

Table (5.17) present a comparison between the error in our methods and other method in [1], where $error = |exact\ value - numerical\ value|$ and $\|err\|_{\infty} = \max|Error|$, with $h=0.01$.

Table 5.17: comparison between the error with reference [1].

x	Error in linear	Error in quadratic	Error obtains in [1]
$\ err\ _{\infty}$	2.220446049250313e-15	2.220446049250313e-15	-

Table (5.18) present a comparison between error obtain using nonpolynomial spline function including linear and polynomial spline function including (1st order and 2nd order: Algorithm (VIE2PS1) and Algorithm (VIE2PS1)) with $h=0.1$.

Table 5.18: Comparison between error obtain using polynomial and nonpolynomial spline functions.

x	Non-polynomial spline		Polynomial spline	
	Error in linear	Error in quadratic	Error in 1 st order	Error in 2 nd order
0	0	0	0	0
0.1	0	0	1.666666666e-02	0
0.2	1.3884260085417e-05	5.551115120e-17	6.666666666e-02	5.551115120e-17
0.3	2.2192613747207e-04	2.220446049e-16	1.500000000e-01	2.220446049e-16
0.4	1.1216304186860e-03	5.551115120e-17	2.666666666e-01	5.551115120e-17
0.5	3.5366466762822e-03	0	4.166666666e-01	0
0.6	8.6085396345747e-03	0	5.999999999e-01	0
0.7	1.7785383032259e-02	4.440892090e-16	8.166666666e-01	4.440892090e-16
0.8	3.2807290948292e-02	8.881784190e-16	1.066666666e+00	8.881784190e-16
0.9	5.5689031157217e-02	8.881784190e-16	1.350000000e+00	8.881784190e-16
1	8.8699894235547e-02	4.440892090e-16	1.666666666e+00	4.440892090e-16
$\ err\ _{\infty}$	8.8699894235547e-02	8.881784190e-16	1.666666666e+00	8.881784190e-16

6. Conclusion.

In this paper, we have introduced numerical methods for approximating the solution of three types of integral equations: which are the VIE's of the 2nd kind, VIE's of the 1st kind and VIE's with weakly singular kernel using linear and quadratic non-polynomial spline functions. Also, we compared our method with polynomial spline of (1st order and 2nd order). So, we get the following:

- The quadratic non-polynomial spline gives better accuracy than linear non-polynomial spline.
- The 2nd order polynomial spline gives better accuracy than 1st order polynomial spline.
- The quadratic non-polynomial spline gives better accuracy than 2nd order polynomial spline.

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